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Basic Topics on Tropical Geometry and Singularities

(トロピカル幾何と特異性に関する基本的トピックス)

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Motivated from real algebraic geometry, Viro, Mikhalkin, Shustin, Itenberg, and other mathematicians have developed the "tropical (algebraic) geometry" [8]. Algebraic curves are tropicalized to piecewise-linear curves. The method was used to construct topological types of real algebraic curves in Hilbert's 16th problem [24].

In this rough sketch, we present several basic topics of tropical geometry, in particular, the notion of hyperfields introduced by Viro recently [25].

【Tropical Limits of Operations】

Let $\mathbf{R}_+$ denote the set of non-negative real numbers.

We fix $h > 0$ and consider the bijection

$$h\log : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$$

defined by

$$u \mapsto h\log u, \quad e^{\frac{x}{h}} \leftarrow x.$$

On $\mathbf{R} \cup \{-\infty\}$, two operations

$$\begin{cases} x +_h y &:= & h\log\left(e^{\frac{x}{h}} + e^{\frac{y}{h}}\right) \\ x \times_h y &:= & h\log\left(e^{\frac{x}{h}} \cdot e^{\frac{y}{h}}\right) = x + y \end{cases}$$

are induced from the summension and the multiplication on $\mathbf{R}_+$.

Set $m = \max\{x,y\}$. Then we have

$$h\log\left(e^{\frac{m}{h}}\right) \leq x +_h y \leq h\log\left(e^{\frac{m}{h}} + e^{\frac{m}{h}}\right),$$

namely,

$$m \leq x +_h y \leq m + h\log 2.$$

Therefore we have that

$$\lim_{h\downarrow 0}(x +_h y) = \max\{x,y\}.$$

【 Tropical Semi-Ring 】

$\mathbf{R}_{\text{trop}} = \mathbf{R} \cup \{-\infty\}$ with the two operations

$$“x + y” := \max\{x,y\}, \quad “x \cdot y” := x + y,$$

is called the *tropical semi-ring* (or the *max-plus algebra*).

Moreover we set $“x/y” = x - y$ if $y \neq -\infty$. Note that there is no tropical subtraction. The tropical sum is idempotent :

$$“x + x” = x,$$

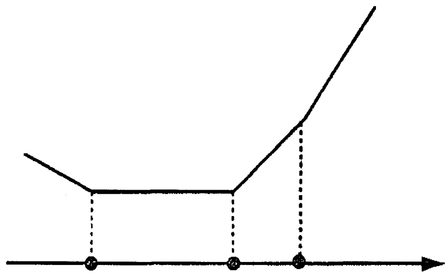
$-\infty$ being the tropical zero.

【 Tropical Polynomials 】

For a finite subset $A \subset \mathbf{Z}^n$, consider a “tropical” (Laurent) polynomial

$$F(x) = “\sum_{j\in A} c_j x^j” = \max\{c_j + j\cdot x \mid j \in A\},$$

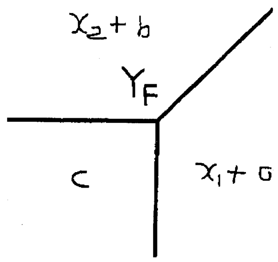
($c_j \in \mathbf{R}$), which is a PL-function on $\mathbf{R}^n$. Then the tropical hypersurface $Y_F \subset \mathbf{R}^n$ is defined by F as the *corner locus* of F.

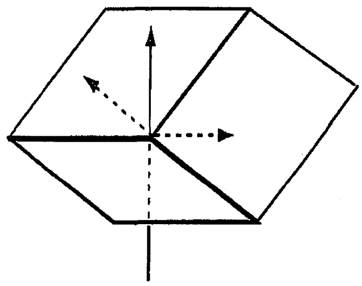


Example 1. (tropical line). We consider

$$F(x_1, x_2) = \text{“}ax_1 + bx_2 + c\text{”} = \max\{x_1 + a,\ x_2 + b,\ c\}.$$

Then Y_F consists of three half-lines meeting at one point.





【 Tropical Hyperfields 】

We define a *multi-valued* addition $\curlyvee$ on $\mathbf{R} \cup \{-\infty\}$: For $a, b \in \mathbf{R} \cup \{-\infty\}$, we set

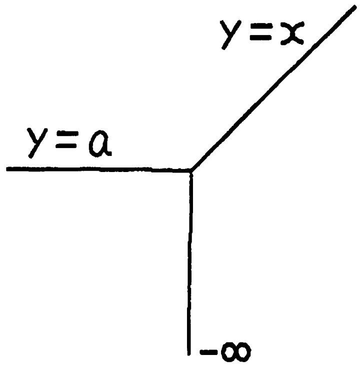
$$a \curlyvee b := \begin{cases} \max\{a, b\}, & (a \neq b) \\ \{y \in \mathbf{R} \cup \{-\infty\} \mid y \leq a\}, & (a = b) \end{cases}$$

The multiplication is defined by the ordinary addition.
We set $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \curlyvee, +)$ and we call it the *tropical hyperfield.*

This implies the natural definition of “tropical zero”.

Example 2. For $a \in \mathbf{R}$, we define the function $x \curlyvee a : \mathbb{Y} \to \mathbb{Y}$. Then we have

$$\text{graph}(x \curlyvee a) = \{y = a, x < a\} \cup \{y = x, a < x\} \cup \{y \leq a, x = a\}.$$



【 Definition of Hyperfields 】

Suppose there are given, on a set X, a multi-valued binary operation $\top$ and a single-valued binary operation $\cdot$.

Then $(X, \top, \cdot)$ is called a *hyperfield* if

- $a \top b = b \top a, \quad a \top (b \top c) = (a \top b) \top c$
- $\exists 0 \in X,\ 0 \top a = a$, for any $a \in X$,
- $\forall a \in X, \exists_1\ -a \in X$ (minus a) such that $0 \in a \top (-a)$.
- $c \in a \top b \iff (-c) \in (-a) \top (-b)$
- The operation $\cdot$ is commutative, associative and
  $0 \cdot a = 0$ holds for any $a \in X$,
- $(X \setminus \{0\}, \cdot\ )$ is a commutative group, which will be denoted by $X^\times$,
- the "distributive law" holds:
  $a \cdot (b \top c) = (a \cdot b) \top (a \cdot c), \quad (b \top c) \cdot a = (b \cdot a) \top (c \cdot a).$

Lemma 3. *The tropical hyperfield* $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \curlyvee, +)$ *is a hyperfield.*

In fact we have

- The zero-element is $-\infty$.
- For $a \in \mathbb{Y}$, $-a$ equals a, since $-\infty \in a \curlyvee b \Leftrightarrow b = a$.
- The commutative group $\mathbb{Y}^\times = (\mathbf{R}, +)$, the unit being $0 \in \mathbf{R}$.

[Tropical Hypersurfaces and Newton Polyhedra]

For a tropical Laurent polynomial $F(x) = $ " $\sum_{j\in A} c_j x^j$ ", we define

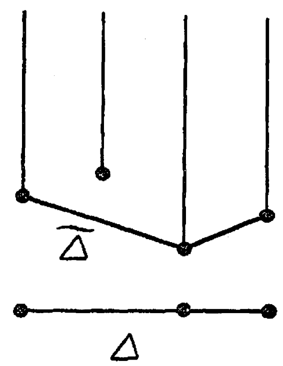
$$v = -c : A \to \mathbf{R}$$

by $v(j) := -c_j, (j \in A)$. Then we set

$$\sqcup(v) := \text{convex hull } \{(j,y) \in \mathbf{R}^n \times \mathbf{R} \mid j \in A, y \geqq v(j)\} \subset \mathbf{R}^{n+1}$$

We set $\Delta = \Delta_F = $ convex hull $(A) \subset \mathbf{R}^n$, and $\widetilde{\Delta}$ the union of compact faces of $\sqcup(v)$. We call $\Delta = \Delta_F$ the Newton polyhedra of F.

Then $\widetilde{\Delta}$ projects to Δ in bijection by $\pi : \widetilde{\Delta} \to \mathbf{R}^n$, $\pi(j,y) := j$. An integral subdivision of Δ is induced from $\widetilde{\Delta}$. We obtain the convex function $\overline{v} : \Delta \to \mathbf{R}$ having $\widetilde{\Delta}$ as its graph.



The tropical hypersurface Y_F is an $(n-1)$-dimensional regular polyhedral complex. (Regularity condition: the boundary of each i-cell is a union of $(i-1)$-cells.)

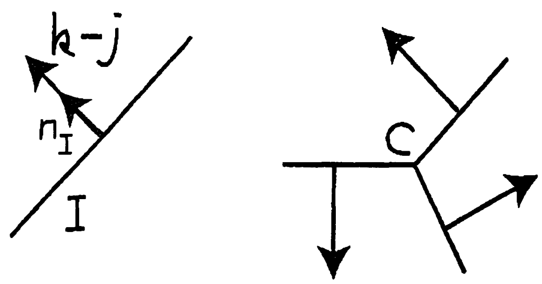
Along each $(n-1)$-cell I, two functions $c_j + j \cdot x, c_k + k \cdot x$ have the same value. From $c_j + j \cdot x = c_k + k \cdot x$, we have the equation

$$(k-j)x + (c_k - c_j) = 0$$

of the hyperplane containing I. Then the integer vector $k - j$ is orthogonal to I. Then there exist the unique positive integer w_I and the primitive integer vector n_I such that $k - j = w_I n_I$.

For each $(n-2)$-cell C, and $(n-1)$-cells $I_1, I_2, \ldots, I_m$ adjacent to C, if we fix a co-orientation of C and take primitive orthogonal vectors n_{I_j}, then we have the *balanced condition*

$$w_{I_1} n_{I_1} + w_{I_2} n_{I_2} + \cdots + w_{I_m} n_{I_m} = 0.$$



Thus the tropical hypersurface Y is an $(n-1)$-dimensional weighted rational polyhedral complex satisfying the regularity condition and the balanced condition.

Tropical hypersurface Y_F is invariant under the deformations, called the *fundamental deformations*, of the tropical Laurent polynomial F.

(1) Replace c by $c' : A \to \mathbf{R}, c'(j) = c(j) + \text{const.}$.

(2) Replace A by $A' = A + j_0, j_0 \in \mathbf{Z}^n$ and c by $c' : A' \to \mathbf{R}$, $c'(j + j_0) = c(j)$.

(3) Replace $c : A \to \mathbf{R}$ by $c' : A' \to \mathbf{R}$ such that convex hull $A' = \Delta$ and the convex function $\overline{-c'} = \overline{-c}$.

【 Legendre Transformations 】

Consider the contact manifold $M = \mathbf{R}^{2n+1}$ with coordinates

$$(x, y, p) = (x_1, \ldots, x_n, y, p_1, \ldots, p_n)$$

and with the contact form $\theta = dy - \sum_{i=1}^{n} p_i dx_i$.

Note that $-\theta = d(\sum_{i=1}^{n} p_i x_i - y) - \sum_{i=1}^{n} x_i dp_i$. Then we have the *double Legendrian fibration*:

$$\mathbf{R}^{n+1} \xleftarrow{\pi_1} \mathbf{R}^{2n+1} \xrightarrow{\pi_2} \mathbf{R}^{n+1},$$

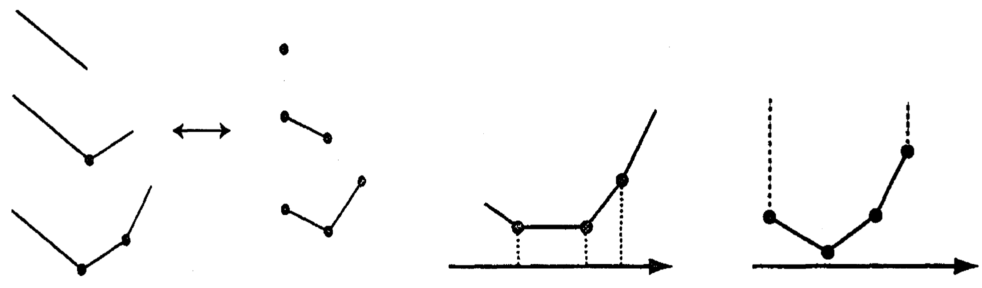
$\pi_1(x, y, p) = (x, y), \quad \pi_2(x, y, p) = (\tilde{y}, p), \quad \tilde{y} = \sum_{i=1}^{n} p_i x_i - y.$

For a function $h : \Delta \to \mathbf{R}$ on a convex set $\Delta \subset \mathbf{R}^n$, the *Legendre transformation* of h is defined as the set of *supporting hyperplanes* of the *epi-graph* of h.

Lemma 4. *The graph of tropical polynomial function*

$$F(x) = \text{"} \sum_{j \in A} c_j x^j \text{"}$$

and the graph of the convex function $\overline{-c} : \mathbf{R}^n \to \mathbf{R}$ *are the Legendre transformations to each other.*



We consider the topological classification problem of tropical polynomial functions preserving corner loci.

Definition 5. Two tropical polynomials $F(x)$ and $G(x)$ are called *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ and $\Psi : \mathbf{R} \to \mathbf{R}$ such that

$$\Psi(F(x)) = G(\Phi(x)), \; \Phi(Y_F) = Y_G.$$

Proposition 6. *There exists a semialgebraic set* $\Sigma \subset \mathbf{R}^A$ *of codim* > 0 *such that, for any* $c \in \mathbf{R}^A \setminus \Sigma$, *the decomposition of* Δ *is simplicial.*

For each connected component U *of* $\mathbf{R}^A \setminus \Sigma$, *the family* $F_c(x), c \in U$ *of tropical polynomial functions is topologically trivial.*

【 Topological Bifurcations of Singularities 】

The topology of a tropical polynomial with a non-simplicial decomposition bifurcates into a generic tropical polynomial.

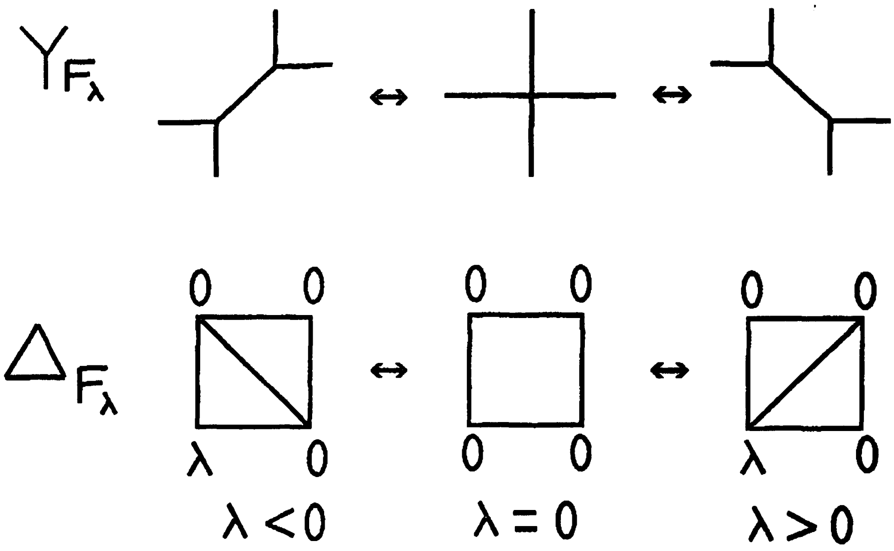
Example 7. Let us consider the tropical polynomial

$$F = \text{" } 0 + 0x_1 + 0x_2 + 0x_1x_2 \text{ "} = \max\{0, x_1, x_2, x_1 + x_2\}.$$

Then F has the deformation:

$$F_\lambda = \text{"}\lambda + 0x_1 + 0x_2 + 0x_1x_2\text{"} = \max\{\lambda, x_1, x_2, x_1 + x_2\}, (\lambda \in \mathbf{R}, \lambda \neq 0).$$

The tropical curve Y_F bifurcates into $Y_{F_\lambda}(\lambda > 0, \lambda < 0)$. The decomposition of Newton polyhedron Δ_F bifurcates into $\Delta_{F_\lambda}(\lambda > 0, \lambda < 0)$.



【 Amoeba and Pachworking 】

For a complex Laurent polynomial

$$f(z) = \sum_{j \in A} b_j z^j \in \mathbf{C}[z_1^{\pm}, \dots, z_n^{\pm}], \quad b_j \in \mathbf{C}^\times,$$

we have a hypersurface

$$Z_f = \{z \in (\mathbf{C}^\times)^n \mid f(z) = 0\} \subset (\mathbf{C}^\times)^n$$

in the complex torus $(\mathbf{C}^\times)^n$.

For a given function $v : A \to \mathbf{R}$, consider the family of polynomials,

$$f_t = f_t^v(z) := \sum_{j \in A} b_j t^{-v(j)} z^j, \quad (t > 0).$$

We call it the *patchworking polynomial* induced by f and v. Note that $f_1 = f$.

Let us define $\mathrm{Log}_t : \mathbf{C}^n \to (\mathbf{R} \cup \{-\infty\})^n$ by

$$\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t |z_1|, \ldots, \log_t |z_n|).$$

We set $\mathcal{A}_f = \mathrm{Log}(Z_f) \subset \mathbf{R}^n$ and we call it the *amoeba* of Z_f.

Proposition 8. (Viro, Kapranov)

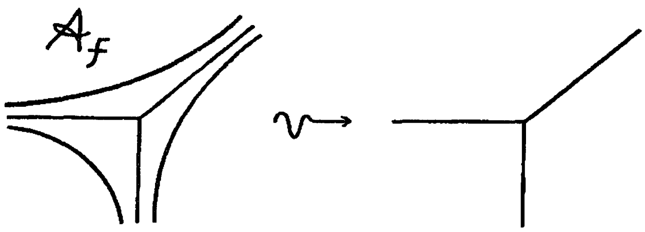
$$\lim_{t \to \infty} \text{Hausdorff-dist}(\mathrm{Log}_t(Z_{f_t}), Y_{f^v_{\text{trop}}}) = 0$$

where

$$f^v_{\text{trop}}(x) := \text{“} \sum_{j \in A} (-v(j)) x^j \text{”} = \max_{j \in A} (j \cdot x - v(j))$$

(Legendre transformation of v).

Example 9. *Amoeba of $f(z_1, z_2) = z_1 + z_2 + 1$.*



【Puiseux Series and Non-Archimedean Amoeba】

Let us denote by $\mathbf{C}[\mathbf{R}]$ the group algebra of the additive group $\mathbf{R}$ over $\mathbf{C}$. We consider its formal version:

A *Puiseux-Laurent series* of real power (Hahn series[4]) is given by

$$a = a(s) = \sum_{p \in I} \alpha_p s^p$$

where $\alpha_p \in \mathbf{C}^\times$ and the support $I = I_a \subset \mathbf{R}$ of a is a well-ordered subset.

We set

$$\mathbf{C}((\mathbf{R})) := \{a(s) \mid a(s) : \text{Puiseux-Laurent series of real power}\} \cup \{0\}.$$

Lemma 10. $\mathbf{K} = \mathbf{C}((\mathbf{R}))$ *is an algebraically closed field.*

Define the valuation val : $\mathbf{C}((\mathbf{R})) \to \mathbf{R} \cup \{\infty\}$ on $\mathbf{C}((\mathbf{R}))$ by

$$\mathrm{val}(a) := \min I_a \in \mathbf{R}, \ (a \in \mathbf{C}((\mathbf{R})) \setminus \{0\}), \quad \mathrm{val}(0) = \infty,$$

Then we have that $\mathrm{val}(a) = \infty$ if and only if $a = 0$, and that

$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b), \quad \mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}.$$

We define the *non-Archimedes norm* on $\mathbf{C}((\mathbf{R}))$ by

$$\|a\| := e^{-\mathrm{val}(a)} \quad (a \in \mathbf{C}((\mathbf{R}))^{\times}), \qquad \|0\| = 0.$$

Then we have the *tropical triangular inequality*

$$\|a+b\| \ \leq \ \max\{\|a\|, \|b\|\} \ = \ \text{“} \ \|a\| + \|b\| \ \text{”}$$

Define Log : $\mathbf{C}((\mathbf{R}))^n \to (\mathbf{R} \cup \{-\infty\})^n$ by

$$\begin{aligned} \mathrm{Log}(a_1, \ldots, a_n) &:= (\log \|a_1\|, \ldots, \log \|a_n\|) \\ &= (-\mathrm{val}(a_1), \ldots, -\mathrm{val}(a_n)). \end{aligned}$$

Given a Laurent polynomial $f(z) = \sum_j a_j z^j \in \mathbf{K}[z, z^{-1}]$, we define

$$Z_f := \{z \in (\mathbf{K}^{\times})^n \mid f(z) = 0\} \subset (\mathbf{K}^{\times})^n.$$

Its Log-image $\mathbf{A}_f := \mathrm{Log}(Z_f) \subset (\mathbf{R} \cup \{-\infty\})^n$ is called the *non-Archimedean amoeba* of Z_f.

Define a tropical Laurent polynomial

$$\begin{aligned} f_{\mathrm{trop}}(x) &:= \text{“} \textstyle\sum_{j \in A} \log \|a_j\| x^j \text{”} = \text{“} \textstyle\sum_{j \in A} (-\mathrm{val}(a_j)) x^j \text{”} \\ &= \max_{j \in A} (j \cdot x - \mathrm{val}(a_j)). \end{aligned}$$

We call $f_{\mathrm{trop}}(x)$ the *tropicalization* of $f(z)$.

Proposition 11. (Kapranov) *Non-Archimedean amoeba is a tropical hypersurface: We have* $\mathbf{A}_f = Y_{f_{\mathrm{trop}}}$.

【 Triangle hyperfield 】

On $\mathbf{R}_+$, define the multi-valued addition

$$\begin{aligned} a\nabla b &:= \{c \in \mathbf{R}_+ \mid |a-b| \leq c \leq a+b\} \\ &= \{|z+w| \mid |z|=a, |w|=b\}. \end{aligned}$$

This reminds us the superposition of waves.

Then $\mathbf{R}_+^{\mathrm{tri}} = (\mathbf{R}_+, \nabla, \cdot)$ is a hyperfield.

【 Amoeba hyperfield 】

By the bijection $\log : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$, we have the hyperfield

$$\log(\mathbf{R}_+^{\mathrm{tri}}) := (\mathbf{R} \cup \{-\infty\},\ \curlyvee\ , +),$$

which is called the *amoeba hyperfield*:

$$a \curlyvee b := \{c \in \mathbf{R} \cup \{-\infty\} \mid \log(|e^a - e^b|) \leq c \leq \log(e^a + e^b)\}.$$

【 Tropical Limits of Amoeba Hyperfield 】

Define, on $\mathbf{R} \cup \{-\infty\}$,

$$\begin{aligned} a \curlyvee_h b &:= h(\tfrac{a}{h} \curlyvee \tfrac{b}{h}) \\ &= \{c \in \mathbf{R} \cup \{-\infty\} \mid \\ &\qquad h\log(|e^{\frac{a}{h}} - e^{\frac{b}{h}}|) \leq c \leq h\log(e^{\frac{a}{h}} + e^{\frac{b}{h}}),\} \\ a \curlyvee_h b &= \{c \in \mathbf{R} \cup \{-\infty\} \mid -\infty \leq c \leq a + h\log 2\} \\ &= [-\infty, a] =: a \curlyvee a. \end{aligned}$$

If $a \neq b$, then $a \curlyvee_h b \to \{\ \max\{a,b\}\ \}$.

$$\lim_{h\to 0} a \curlyvee_h b = a \curlyvee b,$$

$$\lim_{h\to 0} \log(\mathbf{R}_+^{\mathrm{tri}})_h \to \mathbb{Y} \text{ (: tropical hyperfield)}.$$

【 Complex Tropical hyperfield 】

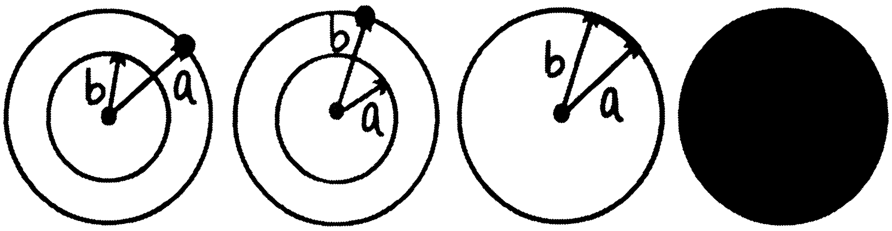
We define a multi-valued addition $\smile$ on $\mathbf{C}$: Let $a,b \in \mathbf{C}$. If $|a| \neq |b|$, then we set $a \smile b := a$ if $|a| > |b|$, and $a \smile b := b$ if $|a| < |b|$.

Suppose $|a| = |b|$. If $b \neq -a$, then

$a \smile b$:= [the shortest arc connecting a and b on the circle $\{z \in \mathbf{C} \mid |z| = |a|\}$].

If $b = -a$, then set

$$a \smile b := \{z \in \mathbf{C} \mid |z| \leq |a|\}.$$



We define the *complex tropical hyperfield* by

$$\mathcal{T}\mathbf{C} := (\mathbf{C}, \smile, \text{the usual multiplication}).$$

On $\mathbf{C}$, we consider the bijection $S_h : \mathbf{C} \to \mathbf{C}$ defined by

$$S_h(z) := \begin{cases} |z|^{\frac{1}{h}} \frac{z}{|z|} & (z \neq 0), \\ 0 & (z = 0). \end{cases}$$

and we define

$$z +_h w := S_h^{-1}(S_h(z) + S_h(w)).$$

Then we have a family of fields $(\mathbf{C}, +_h, \times)$, $h > 0$.

Theorem 12. (Viro [25]) *Let*

$$\Gamma = \{(z, w, z +_h w, h) \in \mathbf{C}^3 \times \mathbf{R}_+ \mid (z, w, h) \in \mathbf{C}^2 \times \mathbf{R}_+\}.$$

Then

$$\overline{\Gamma} \cap (\mathbf{C}^3 \times \{0\}) = \{(a,\ b,\ a \smile b,\ 0) \mid (a, b) \in \mathbf{C}^2\}.$$

【Viro's Diagram 2010】

Thus we have the diagram:

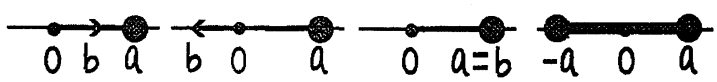
$$\begin{array}{ccc}
\text{Complex Algebraic Geometry} & & \text{Complex Tropical Geometry} \\
\mathbf{C} \cong \mathbf{C}_h & \xrightarrow{h\to 0} & \mathbf{C}_0 = \mathcal{T}\mathbf{C} \\
{\scriptstyle z\mapsto |z|}\Big\downarrow & & \Big\downarrow{\scriptstyle z\mapsto |z|} \\
\mathbf{R}_+^{\mathrm{tri}} \cong \mathbf{R}_{+h}^{\mathrm{tri}} & \xrightarrow{h\to 0} & \mathbf{R}_{+0}^{\mathrm{tri}} = \mathbb{Y}_\times \\
{\scriptstyle x\mapsto \log x}\Big\downarrow & & \Big\downarrow{\scriptstyle x\mapsto \log x} \\
\log(\mathbf{R}_+^{\mathrm{tri}}) \cong \log(\mathbf{R}_+^{\mathrm{tri}})_h & \xrightarrow{h\to 0} & \mathbb{Y} \\
\text{Amoeba Geometry} & & \text{Tropical Geometry}
\end{array}$$

【 Real Tropical Hyperfield 】

Question: What is the real counterpart of the complex tropical hyperfield ?

We are naturally led to define the multi-valued addition $\smile_{\mathbf{R}}$ on $\mathbf{R}$ induced from $\smile$ on $\mathbf{C}$: For $a,b \in \mathbf{R}$, we set

$$\begin{cases} a\smile_{\mathrm{R}} b & := \quad a \quad \text{if } |a| > |b|, \\ a\smile_{\mathrm{R}} b & := \quad b \quad \text{if } |a| < |b|, \\ a\smile_{\mathrm{R}} a & := \quad a, \\ a\smile_{\mathrm{R}}(-a) & := \quad [-a,a]. \end{cases}$$



Theorem 13. $(\mathrm{R}, \smile_{\mathrm{R}}, \times)$ *is a hyperfield. Moreover let*

$$\Gamma_{\mathbf{R}} = \{(a,b,a+_h b,h) \in \mathbf{R}^3 \times \mathbf{R}_+ \mid (a,b,h) \in \mathbf{R}^2 \times \mathbf{R}_+\}.$$

Then we have

$$\overline{\Gamma_{\mathbf{R}}} \cap (\mathbf{R}^3 \times \{0\}) = \{(a,\ b,\ a\smile_{\mathbf{R}} b,\ 0) \mid (a,b) \in \mathbf{R}^2\}.$$

The real tropical hyperfield is, in some sense, a "double covering" of the tropical hyperfield via $x \mapsto \log|x|$. Therefore, "real tropical geometry" can be constructed as a "double covering" of tropical geometry.

【 Several Questions 】

Question: Is the complex tropical hyperfield $\mathcal{T}\mathbf{C}$ is algebraically closed, in an appropriate sense ?

Question: Are the real tropical hyperfield and the tropical hyperfield $\mathbb{Y}$ real closed ?

Question: What is the real tropical algebraic geometry ?

In Amoeba geometry, it is known the *Ronkin function*

$$N_f(x) := \frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathrm{Log}^{-1}(x)} \log|f(z)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

is linear on each connected component of $\mathbf{R}^n \setminus \mathcal{A}_f$. We have $\mathrm{grad} N_f : \mathbf{R}^n \setminus \mathcal{A}_f \to \Delta \cap \mathbf{Z}^n$ and $\mathrm{grad} N_f$ separates every connected components of $\mathbf{R}^n \setminus \mathcal{A}_f$.

Question: Can the Ronkin function be described in terms of the amoeba hyperfield ?

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